Monge solutions of time-dependent Hamilton-Jacobi equations in metric spaces

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Abstract

In this talk, we introduce a notion of Monge solutions for time-dependent Hamilton-Jacobi equations in metric spaces. The key idea is to reformulate the equation as a stationary problem under the assumption of Lipschitz regularity for the initial data. We establish the uniqueness and existence of bounded Lipschitz Monge solutions to the initial value problem and discuss their equivalence with existing notions of metric viscosity solutions.

1 Introduction

The notion of viscosity solutions is a well-established framework for studying well-posedness of Hamilton–Jacobi equations in Euclidean spaces. In recent years, increasing attention has been devoted to extending the Hamilton–Jacobi equation to general metric spaces. This direction is motivated both by intrinsic mathematical interest and by applications where the underlying domain is naturally non-smooth or non-linear, such as dynamics on networks and graphs, and models of traffic flow or front propagation constrained by complex geometries.

From an analytical viewpoint, moving beyond the Euclidean setting raises fundamental questions. Classical tools, such as gradients, smooth test functions, and PDE techniques relying on local coordinates, are no longer directly available. As a result, the formulation of a suitable notion of solution and the proof of existence, uniqueness, and stability require alternative structures that are compatible with the metric geometry. Complete length spaces provide a natural setting for this purpose, while distance-minimizing curves need not exist in full generality; distances are still described as the infimum of curve lengths and can be approximated by almost minimizing curves. This supports a curve-based notion of "directional" information, avoiding reliance on classical derivatives.

To gain further understanding of Hamilton–Jacobi dynamics in this framework, we introduce the notion of Monge solutions on a complete length space $(\mathbf{X}, d)$ for the time-dependent Hamilton-Jacobi equation of the form

$$\partial_t u(x,t) + H(x,t,|\nabla u|(x,t)) = 0, \qquad (x,t) \in \mathbf{X} \times (0,T) \tag{1.1}$$

for $T > 0$ given, along with a bounded continuous initial value

$$u(x,0) = u_0(x), \qquad x \in \mathbf{X}. \tag{1.2}$$

Recall that $(\mathbf{X}, d)$ is called a length space if for all $x, y \in \mathbf{X}$, $d(x,y)$ coincides with the infimum of $\ell(\gamma)$ of all rectifiable curves γ in $\mathbf{X}$ joining x, y, where $\ell(\gamma)$ denotes the length of γ. Here the (local) slope of u with respect to the space variable, denoted by $|\nabla u|(x,t)$, is defined by

$$|\nabla u|(x,t) = \limsup_{y \to x} \frac{|u(x,t) - u(y,t)|}{d(x,y)}.$$

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Two types of metric viscosity solutions have been previously introduced in [4, 7] and in [1, 2, 3], to handle such equations, including the stationary case. In [4, 7], we refer to this as a curve-based solution. The authors define the solution using compositions with curves, transforming the problem into a one-dimensional setting and mimicking the definition of a viscosity solution in the Euclidean setting. On the other hand, we refer to the solution in [3] as a slope-based solution since it defines the solution using (local) slope approaches. Additionally, in the case of the eikonal equation

$$|\nabla u|(x) = f(x) \text{ in } \Omega,$$

where $\Omega \subset \mathbf{X}$ is a domain and f is continuous in Ω, they are proved in [5] to be equivalent in a complete length space. The equivalence is established through a third notion of solutions known as Monge solutions, first introduced in [8] in the Euclidean setting, which offers a convenient definition that does not rely on test functions. In this case, the Monge solution is defined as a locally Lipschitz function u that satisfies

$$|\nabla^- u|(x) = f(x) \qquad \text{for all } x \in \Omega,$$

where the (local) subslope $|\nabla^- u|(x)$ is defined by

$$|\nabla^- u|(x) = \limsup_{y \to x} \frac{\max\{u(x) - u(y), 0\}}{d(x, y)}.$$

The Monge approach to such time-dependent problems has not been available even in Euclidean spaces. In [6], we showed the existence and uniqueness of the Monge solution to the associated initial value problem and the equivalence with the other two notions for certain cases.

2 Main Results

For the well-posedness theory of Monge solutions to the Hamilton-Jacobi equations (1.1)-(1.2), we distinguish two settings: (i) the Eikonal-type case and (ii) the case of a general superlinear Hamiltonian. We conclude by stating the corresponding equivalence results.

2.1 The Eikonal-type case

Let $\mathbf{X}_T = \mathbf{X} \times (0, T)$ and $\mathrm{Lip}_{\mathrm{loc}}(\mathbf{X}_T)$ denote the set of all locally Lipschitz functions in $\mathbf{X}_T$. We first consider the eikonal-type case for the Hamiltonian $H(x, t, p) = p - f(x, t)$ where f is bounded and continuous. Then, (1.1) reads

$$\partial_t u(x, t) + |\nabla u|(x, t) = f(x, t), \qquad (x, t) \in \mathbf{X}_T. \tag{2.1}$$

In order to formulate a new notion of Monge solution for the problem (2.1), our strategy is to treat the time and space variables jointly, reformulating (2.1) as a stationary equation in the product metric space $\mathbf{X} \times (0, T)$. To adapt the method in [5] for stationary eikonal equation to this new formulation, it is desirable to have a nonnegative term of $\partial_t u$ so that the left hand side of (2.1) can be rewritten as $|\partial_t u| + |\nabla u|$, which represents a certain gradient norm of u in space-time. This is achieved by seeking a Lipschitz-in-time solution and a variable change. Therefore, we define our solution space as follows

$$S(\mathbf{X}_T) := \bigcup_{k \geq 0} S_k(\mathbf{X}_T).$$

where for $k \geq 0$,

$$S_k(\mathbf{X}_T) := \big\{u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{X}_T) : u(x,t) - u(x,s) \geq -k(t-s), \\ \text{for all } x \in \mathbf{X}, 0 < s < t < T\big\}.$$

It is easy to see that $S_0(\mathbf{X}_T)$ represents the set of all functions $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{X}_T)$ that are non-decreasing in t. Moreover, any Lipschitz function in $\mathbf{X}_T$ belongs to $S(\mathbf{X}_T)$. For any function $u \in S_k(\mathbf{Y}_T)$ with $k \geq \max\{0, -\inf_{\mathbf{X}_T} f\}$ that satisfies (2.1), the change of variable

$$v(x,t) = u(x,t) + kt \tag{2.2}$$

implies that v will formally satisfies

$$|\partial_t v(x,t)| + |\nabla v|(x,t) = k + f(x,t),$$

whose left-hand side can be understood as a certain gradient in space-time. Therefore, in view of the Monge solution for the Eikonal equations, we can define our notion of Monge solutions $u \in S(\mathbf{X}_T)$ for (2.1) based on the subslope-type value $|D^- v|$ of v in space-time defined as

$$|D^- v|(x,t) = \limsup_{\delta \to 0+} \sup \left\{ \frac{v(x,t) - v(y, t-\delta)}{\delta} : y \in \mathbf{X},\ d(x,y) \leq \delta \right\}. \tag{2.3}$$

Thus, we define the Monge solution for (2.1) as follows.

Definition 2.1 ([6]). A function $u \in S(\mathbf{X}_T)$ is said to be a Monge solution (resp., Monge subsolution, Monge supersolution) of (2.1) if there exists $k \geq \max\{0, -\inf_{\mathbf{X}_T} f\}$ such that $u \in S_k(\mathbf{X}_T)$, and $v \in S_0(\mathbf{X}_T)$ given by (2.2) satisfies

$$|D^- v|(x,t) = f(x,t) + k \quad (\text{resp., } \leq, \geq) \tag{2.4}$$

for any $(x,t) \in \mathbf{X}_T$, where $|D^- v|(x,t)$ is given as in (2.3).

Please note that the Monge solution is independent of the choice of any particular k in the above definition. Below we present the existence and uniqueness of the Monge solution to the initial value problem (2.1)-(1.2).

Theorem 2.2 ([6], Main result 1). *Let $(\mathbf{X}, d)$ be a complete length space. Let $T > 0$. Assume that $u_0 \in \mathrm{Lip}(\mathbf{X})$ is bounded. Assume in addition that $f \in C(\mathbf{X}_T)$ is bounded and $f(x,t)$ is Lipschitz with respect to either x or t. Then, there exists a unique bounded Monge solution $u \in \mathrm{Lip}(\mathbf{X} \times [0,T))$ of (2.1) satisfying (1.2) in the sense of*

$$\sup_{x \in \mathbf{X}} |u(x,t) - u_0(x)| \to 0 \quad \text{as } t \to 0+. \tag{2.5}$$

The requirement for the Monge solution to be in the space $S(\mathbf{X}_T)$ is achieved by demonstrating the Lipschitz property of the solution, which follows from the Lipschitz assumption on the initial data. For the uniqueness, we follow a similar idea to that in [5] by proving the classical comparison principle, which compares the subsolution and supersolution in all space-time, given the condition at the initial time. Moreover, the existence is shown by adapting the classical control-theoretic interpretation for eikonal equations

For time-dependent Hamilton-Jacobi equations other than the eikonal type, our definition of Monge solutions becomes more intricate due to the loss of degree-1 homogeneity. We are still able to extract a quantity to replace the subslope used in the eikonal type case by treating the time-dependent problem as a stationary problem in the space-time product space and using a standard Lagrangian formulation. We first assume that

(H1) $p \mapsto H(x,t,p)$ is convex and nondecreasing in $[0,\infty)$ for any $x \in \mathbf{X}$ and $t \in (0,T)$. It is also superlinear in the sense that

$$\inf \left\{ \frac{H(x,t,p)}{p} : (x,t) \in \mathbf{X}_T,\ p \geq R \right\} \to \infty \quad \text{as } R \to \infty. \tag{2.6}$$

Additionally, let $L : \mathbf{X} \times (0,T) \times [0,\infty) \to \mathbb{R}$ denotes the Lagrangian associated to H, that is,

$$L(x,t,q) = \sup_{p \geq 0} \{pq - H(x,t,p)\} \quad \text{for all } (x,t) \in \mathbf{X}_T,\ q \geq 0. \tag{2.7}$$

We then formulate the definition of Monge solution for a general superlinear Hamiltonian in the following way.

Definition 2.3 ([6]). A function $u \in S(\mathbf{X}_T)$ is said to be a Monge solution (resp., Monge subsolution, Monge supersolution) of (1.1) if there exists $k \geq 0$ such that $u \in S_k(\mathbf{X}_T)$ and $v \in S_0(\mathbf{X}_T)$ given by (2.2) satisfies

$$|D_L^- v|(x,t) = k \quad (\text{resp., } \leq,\ \geq) \tag{2.8}$$

for any $(x,t) \in \mathbf{X}_T$, where $|D_L^- v|(x,t)$ at $(x,t) \in \mathbf{X}_T$ is defined by

$$\begin{aligned} |D_L^- v|(x,t) = \lim_{\delta \to 0+} \sup \Bigg\{ & \frac{v(x,t) - v(y,s)}{t-s} - L\left(x, t, \frac{d(x,y)}{t-s}\right) : \\ & (y,s) \in \mathbf{X}_T,\ t - \delta \leq s < t,\ 0 < d(x,y) \leq \delta \Bigg\}. \end{aligned} \tag{2.9}$$

Analogous existence and uniqueness results can be obtained for (1.1), where $L \in C(\mathbf{X} \times (0,T) \times [0,\infty))$ defined as in (2.7) is assumed to satisfy a set of conditions as listed below. In view of (H1), we see that $q \mapsto L(x,t,q)$ is convex and nondecreasing in $[0,\infty)$. We also obtain

$$H(x,t,p) = \sup_{q \geq 0} \{pq - L(x,t,q)\}, \quad \text{for } (x,t) \in \mathbf{X}_T,\ p \geq 0. \tag{2.10}$$

More assumptions on L are given as follows.

(H2) There exists a convex function $m \in C(\mathbb{R})$ such that

$$\inf_{(x,t) \in \mathbf{X}_T} L(x,t,q) \geq m(q) \quad \text{for all } q \geq 0. \tag{2.11}$$

and

$$\frac{m(q)}{q} \to \infty \quad \text{as } q \to \infty. \tag{2.12}$$

(H3) $\sup_{(x,t) \in \mathbf{X}_T} L(x,t,0) < +\infty$.

(H4) L is locally uniformly continuous in $\mathbf{X} \times (0,T) \times [0,\infty)$, and there exists a modulus of continuity ω_L such that

$$\begin{aligned} |L(x,t,q) - L(y,s,q)| \leq \omega_L(d(x,y) + |t-s|)(1 + |m(q)|) \\ \text{for all } x, y \in \mathbf{X},\ t,s \in (0,T),\ q \geq 0, \end{aligned} \tag{2.13}$$

where $m \in C(\mathbb{R})$ is given in (H2).

(H5) L is Lipschitz continuous with respect to the time variable; that is, there exists $C_T > 0$ such that

$$|L(x,t,q) - L(x,s,q)| \leq C_T|t-s| \quad \text{for all } x \in \mathbf{X},\ t,s \in (0,T),\ q \geq 0. \tag{2.14}$$

The second main result is presented as follows.

Theorem 2.4 ([6], Main result 2)**.** *Let $(\mathbf{X}, d)$ be a complete length space and $T > 0$. Assume that $H \in C(\mathbf{X} \times (0,T) \times [0,\infty))$ satisfies (H1) and $L \in C(\mathbf{X} \times (0,T) \times [0,\infty))$ given by (2.7) satisfies (H2)–(H5). Let $u_0 \in \mathrm{Lip}\,(\mathbf{X})$ be bounded. Then, there exists a unique bounded Monge solution $u \in \mathrm{Lip}\,(\mathbf{X} \times [0,T))$ of (1.1) satisfying (2.5).*

Similar to that of the eikonal-type case, the establishment of the uniqueness and existence of Monge solutions to the associated initial value problem (1.1)-(1.2) due to the comparison principle and a generalization of the classical control-theoretic (variational) interpretation of (1.1), respectively. For both cases, the comparison principle is proved more directly without involving the classical doubling variable method and could be extended to handle discontinuous settings.

In addition to the uniqueness and existence results for Monge solutions, we also discuss the connection with other existing approaches. We focus on the special case when H is independent of t, i.e., $H = H(x,p)$, for which the equation (1.1) reduces to

$$\partial_t u(x,t) + H(x, |\nabla u|(x,t)) = 0, \qquad (x,t) \in \mathbf{X}_T. \tag{2.15}$$

We can compare the three notions in this case, since the well-posedness results for other existing notions of metric viscosity solutions are explicitly available in [7, 3] in this setting. Our equivalence result applies to the initial value problem for the following typical Hamilton-Jacobi equation:

$$\partial_t u(x,t) + a(x)|\nabla u|^\alpha(x,t) = f(x), \qquad (x,t) \in \mathbf{X}_T, \tag{2.16}$$

where $\alpha > 1$ is given, and $a, f : \mathbf{X} \to \mathbb{R}$ are assumed to be bounded uniformly continuous with $\inf_{\mathbf{X}} a > 0$. Here is the last main result.

Theorem 2.5 ([6], Main result 3)**.** *Let $(\mathbf{X}, d)$ be a complete length space and $T > 0$. Let $\alpha > 1$ and $a, f : \mathbf{X} \to \mathbb{R}$ be bounded uniformly continuous functions such that $\inf_{\mathbf{X}} a > 0$. Assume that $u_0 \in \mathrm{Lip}\,(\mathbf{X})$ is bounded and $u \in C(\mathbf{X}_T)$ satisfies (2.5). Then the following statements are equivalent.*

(i) u is a curve-based viscosity solution of (2.16);

(ii) u is a slope-based viscosity solution of (2.16);

(iii) u is a Monge solution of (2.16).

Moreover, we showed that the Monge solution is Lipschitz continuous in $\mathbf{X} \times [0,T)$. As a consequence, we see that the other two notions of metric viscosity solutions of the initial value problem for (2.16) are Lipschitz continuous as well. We refer to [6] for more general results. Here, the equivalent is relying on the initial condition; the future project would be investigating whether the result holds locally, meaning the equivalent holds without knowing the initial datum.

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